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Cayley–Hamilton theorem for 2×2 matrices over the Grassmann algebra

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Abstract

It is shown that the characteristic polynomial of matrices over a Lie nilpotent ring introduced recently by Szigeti is invariant with respect to the conjugation action of the general linear group. Explicit generators of the corresponding algebra of invariants in the case of 2×2 matrices over an algebra over a field of characteristic zero satisfying the identity $[[x,y],z]=0$ are described. In this case the coefficients of the characteristic polynomial are expressed by traces of powers of the matrix, yielding a compact form of the Cayley–Hamilton equation of 2×2 matrices over the Grassmann algebra. ©1998 Elsevier Science B.V. All rights reserved.

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1. Noncommutative Cayley–Hamilton theorem related to invariant theory

Let $\mathbb{Z}\langle x_{ij}\rangle = \mathbb{Z}\langle x_{ij} \mid 1 \le i,j \le n\rangle$ be the free associative algebra over the ring of integers. We think of the variables as entries of a generic matrix $X=(x_{ij})$. In [10] Szigeti constructed for any $m \in \mathbb{N}$ the mth *right characteristic polynomial*

$$\chi^{(m)}(t) = t^d + \sum_{i=1}^{d} f_i(x_{11},\ldots,x_{nn})t^{d-i} \in \mathbb{Z}\langle x_{ij}\rangle[t]$$

of X (t is a commuting indeterminate over $\mathbb{Z}\langle x_{ij}\rangle$). The degree d of this polynomial equals to n^m. We shall recall the exact construction below.

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Let $A=(a_{ij})$ be any $n\times n$ matrix over a ring R. We may substitute x_{ij} by a_{ij} $(1\leq i,j\leq n)$ and get the mth right characteristic polynomial

$$\chi_A^{(m)}(t)=t^d+\sum_{i=1}^{d} f_i(a_{11},\dots,a_{nn})t^{d-i}\in R[t]$$

of A. Define the Lie brackets $[x_1,\dots,x_r]$ recursively by

$$[x_1,x_2]=x_1x_2-x_2x_1,$$

$$[x_1,\dots,x_{r+1}]=[[x_1,\dots,x_r],x_{r+1}] \quad \text{for } r\geq 3.$$

The main result of [10] is that if R satisfies the identity $[x_1,\dots,x_{m+1}]=0$, or in other words, if R is Lie nilpotent of index m, and A is an $n\times n$ matrix over R, then the left substitution of A into its mth right characteristic polynomial is zero, that is,

$$A^d+\sum_{i=1}^{d} A^{d-i}f_i(a_{11},\dots,a_{nn})=0\in R^{n\times n}.$$

Note that the powers of A do not commute with the coefficients $f_i(a_{11},\dots,a_{nn})$, so it is important to take the left substitution here. There is an analogous construction of the mth left characteristic polynomial, and the right substitution of an $n\times n$ matrix over a Lie nilpotent ring of nilpotency index m into its left characteristic polynomial is also zero.

In the sequel we assume that R is an algebra over a field K of characteristic zero. The general linear group $Gl_n=Gl_n(K)$ acts on $R^{n\times n}$ by conjugation. It is natural to expect the characteristic polynomial to be invariant with respect to these automorphisms of $R^{n\times n}$, so we slightly modify the construction of the characteristic polynomial from [10].

Consider the natural homomorphism

$$\phi : K\langle x_{ij}\rangle \to K[x_{ij}]$$

onto the commutative polynomial algebra. The map

$$y_1\dots y_r \to \frac{1}{r!}\sum_{\pi\in Sym(r)} y_{\pi(1)}\cdots y_{\pi(r)}$$

from the set of monomials of $K[x_{ij}]$ to $K\langle x_{ij}\rangle$ extends linearly to a map

$$\sigma : K[x_{ij}]\to K\langle x_{ij}\rangle,$$

and clearly we have $\phi\circ\sigma=id_{K[x_{ij}]}$. The symmetric group of degree r acts on the right on the degree r homogeneous component of $K\langle x_{ij}\rangle$ by place permutation, namely, $z_1\dots z_r\cdot\pi=z_{\pi(1)}\dots z_{\pi(r)}$ for any monomial $z_1\dots z_r\in K\langle x_{ij}\rangle$ and $\pi\in Sym(r)$. Obviously, for any $f\in K[x_{ij}]$ the polynomial $\sigma(f)$ is the unique symmetric preimage of f, that is, $\sigma(f)$ is the unique element in $\phi^{-1}(f)$ whose degree r homogeneous component is fixed by $Sym(r)$ for all r.

The maps ϕ and σ extend naturally to maps between $K\langle x_{ij}\rangle^{n\times n}$ and $K[x_{ij}]^{n\times n}$, and we denote these maps by the same symbols ϕ,σ. We define the *adjoint of X* as follows:

$$adj(X) = \sigma(adj(\phi(X))),$$

where $adj(\phi(X))$ denotes the ordinary adjoint of the matrix $\phi(X)$ with commuting entries. (The (i,j) entry of $adj(\phi(X))$ is the determinant of the $(n-1)\times(n-1)$ minor of $\phi(X)$ obtained by removing the jth row and the ith column, multiplied by $(-1)^{i+j}$.) The adjoint of an arbitrary matrix $(a_{ij}) = A \in R^{n\times n}$ is obtained by substituting $x_{ij}\to a_{ij}$ $(1\le i,j\le n)$ in each element of $adj(X)$, so

$$adj(A) = adj(X)|_{X\to A}.$$

Define $X_0, X_1, \dots, X_m$ recursively by $X_0 = X = (x_{ij}) \in K\langle x_{ij}\rangle^{n\times n}$, $X_1 = adj(X_0)$, and $X_{i+1} = adj(XX_1\dots X_i)$ for $i = 1,\dots,m-1$. The mth *right determinant of X* is defined by

$$det^{(m)}(X) = \frac{1}{n}Tr(XX_1\dots X_m),$$

where $Tr(-)$ denotes the usual *trace* function, that is, the sum of the diagonal entries of the matrix. By the results of [10] we have that

$$XX_1\dots X_m = det^{(m)}(X)I + B,$$

where I denotes the $n\times n$ unit matrix, each element of B is contained in the mth Lie bracket ideal of $K\langle x_{ij}\rangle$ (i.e., the ideal generated by elements of the form $[u_1,\dots,u_{m+1}]$, $u_s\in K\langle x_{ij}\rangle$) and the trace of B equals to zero. If we add the $(1,1)$ entry of B to our mth right determinant, then we get the determinant from [10]. In particular, these determinants are the same modulo the mth Lie bracket ideal, and all the results of [10] remain true with the same proof for our determinant.

The mth right determinant of an arbitrary matrix $A\in R^{n\times n}$ is defined by

$$det^{(m)}(A) = det^{(m)}(X)|_{X\to A}.$$

Now the mth characteristic polynomial can be defined as in the commutative case:

$$\chi^{(m)}(t) = det^{(m)}(tI - X),$$

where t is a commuting indeterminate over $K\langle x_{ij}\rangle$, and

$$\chi_A^{(m)}(t) = \chi^{(m)}(t)|_{X\to A} = det^{(m)}(tI - A)$$

for an arbitrary matrix A. For any $g\in Gl_n$ the map

$$x_{ij}\to \text{“the } (i,j) \text{ entry of } gXg^{-1}\text{”}$$

induces a linear transformation on $Span_K\{x_{ij}\mid 1\le i,j\le n\}$. This representation of Gl_n extends diagonally to an action on $K\langle x_{ij}\rangle = K\langle x_{ij}\mid 1\le i,j\le n\rangle$.

Theorem 1.1. *The coefficients of the mth characteristic polynomial are invariant under the action of Gl_n by conjugation, that is,*

$$f_k(x_{11},\ldots,x_{nn}) \in K\langle x_{ij}\rangle^{Gl_n} = \{f \in K\langle x_{ij}\rangle \mid g\cdot f = f \ \forall g \in Gl_n\}$$

for $k = 1,\ldots,d$.

Proof. First we claim that

$$adj(gXg^{-1}) = g\cdot adj(X)\cdot g^{-1}$$

for any $g \in Gl_n$. Since the homomorphism $\phi : K\langle x_{ij}\rangle^{n\times n} \to K[x_{ij}]^{n\times n}$ commutes with the conjugation action of Gl_n, and conjugation commutes with forming the adjoint in the commutative case, and the natural homomorphism $K\langle x_{ij}\rangle \to K[x_{ij}]$ is Gl_{n^2} equivariant (where Gl_{n^2} acts on both algebras by linear substitution of the variables), we have the chain of equalities

$$\begin{aligned}\phi(g\cdot adj(X)\cdot g^{-1}) &= g\phi(adj(X))g^{-1} = g\cdot adj(\phi(X))\cdot g^{-1}\\ &= adj(g\phi(X)g^{-1}) = \phi(adj(X))|_{\phi(X)\to g\phi(X)g^{-1}}\\ &= \phi(adj(X)|_{X\to gXg^{-1}}) = \phi(adj(gXg^{-1})),\end{aligned}$$

showing that $g\cdot adj(X)\cdot g^{-1}$ and $adj(gXg^{-1})$ have the same images under ϕ. On the other hand, each entry of both of them is homogeneous of degree $n-1$, so it suffices to show that each entry of both of them is fixed by the action of $Sym(n-1)$ by place permutation. This is true for the entries of $adj(X)$ by definition, and the entries of $g\cdot adj(X)\cdot g^{-1}$ are linear combinations of the entries of $adj(X)$. The action of Gl_n restricted to the homogeneous component of degree $n-1$ of $K\langle x_{ij}\rangle$ commutes with the action of $Sym(n-1)$. Hence the entries of $adj(X)|_{X\to gXg^{-1}}$ are also symmetric, and the claim follows.

Therefore the chain of equalities

$$\begin{aligned}adj(X)|_{X\to gAg^{-1}} &= (adj(X)|_{X\to gXg^{-1}})|_{X\to A} = adj(gXg^{-1})|_{X\to A}\\ &= (g\cdot adj(X)\cdot g^{-1})|_{X\to A} = g(adj(X)|_{X\to A})g^{-1}\end{aligned}$$

shows that

$$adj(gAg^{-1}) = g\cdot adj(A)\cdot g^{-1}$$

for any matrix A. Hence $X_i|_{X\to gXg^{-1}} = gX_ig^{-1}$, and since the trace is invariant with respect to conjugation, we have that

$$det^{(m)}(gXg^{-1}) = \frac{1}{n}Tr(gXX_1\ldots X_mg^{-1}) = \frac{1}{n}Tr(XX_1\ldots X_m) = det^{(m)}(X)$$

for any $g \in Gl_n$. Obviously, this implies that

$$det^{(m)}(gAg^{-1}) = det^{(m)}(A)$$

for an arbitrary matrix A. Finally, we conclude that

$$\chi^{(m)}_{gXg^{-1}}(t) = det^{(m)}(tI - gXg^{-1}) = det^{(m)}(g(tI - X)g^{-1})$$

$$= det^{(m)}(tI - X) = \chi^{(m)}(t),$$

or in other words, $f_k(x_{11},\ldots,x_{nn}) \in K\langle x_{ij}\rangle^{Gl_n}$ for $k = 1,\ldots,n^m$. $\square$

The mth characteristic polynomial has relevance for matrices over K-algebras that are Lie nilpotent of index m, because in this case we have a Cayley–Hamilton theorem. Working with matrices whose entries are taken from a K-algebra satisfying the identity $[x_1,\ldots,x_{m+1}] = 0$ it is more natural to consider the coefficients of the characteristic polynomial of a generic matrix as elements of the relatively free algebra defined by this identity. More precisely, let I be the ideal of $K\langle x_{ij}\rangle$ generated by $[u_0, u_1,\ldots,u_m]$ as u_k vary in $K\langle x_{ij}\rangle$ (i.e., I is the T-ideal generated by the mth Lie bracket). We have the natural onto homomorphism

$$K\langle x_{ij}\rangle \to K\langle x_{ij}\rangle / I.$$

We do not want to introduce new letters to denote the image of x_{ij}. From now on we think of x_{ij} $(1 \le i,j \le n)$ as generators of the relatively free algebra $K\langle x_{ij}\rangle/I$, and the entries of X, $adj(X)$ and the coefficients of $\chi^{(m)}(t)$ are the corresponding elements in $K\langle x_{ij}\rangle/I$. For example, from now on we consider the mth characteristic polynomial $\chi^{(m)}(t)$ as an element of $(K\langle x_{ij}\rangle/I)[t]$. As we noted earlier $\chi^{(m)}(t) \in (K\langle x_{ij}\rangle/I)[t]$ is the same as the corresponding characteristic polynomial in [10]. The action of Gl_n on the free algebra $K\langle x_{ij}\rangle$ induces an action on $K\langle x_{ij}\rangle/I$, because I is stable with respect to the action of Gl_n. Note that by linear reductivity of Gl_n the natural homomorphism $K\langle x_{ij}\rangle \to K\langle x_{ij}\rangle/I$ restricted to the corresponding algebras of invariants is also surjective. Theorem 1.1 has the following immediate corollary.

Corollary 1.2. *The coefficients of the mth characteristic polynomial lie in* $(K\langle x_{ij}\rangle/I)^{Gl_n}$.

Remark. In the commutative case the coefficients of the characteristic polynomial generate $K[x_{ij}]^{Gl_n}$. This fact motivates the question whether an analogous result holds for $(K\langle x_{ij}\rangle/I)^{Gl_n}$. It was shown in [4] that R^G is a finitely generated algebra for any rational action of a reductive algebraic group G on a finitely generated algebra R, if R satisfies the identity $[x_1,\ldots,x_{m+1}] = 0$. Hence the algebra $(K\langle x_{ij}\rangle/I)^{Gl_n}$ is at least finitely generated. The computation in the next section is an illustrating example for the theory developed in [4].

2. Explicit basic invariants in the case $n=m=2$

The mth characteristic polynomial was invented in [10] in order to study the polynomial identities of matrices over the infinite dimensional Grassmann algebra

$$E=K\langle v_1,v_2,\dots \mid v_iv_j+v_jv_i=0\ \forall i,j\rangle.$$

Identities of $E^{n\times n}$ deserve attention because of their distinguished role in Kemer's classification of T-ideals (see [8]). The algebra E is Lie nilpotent of index 2, moreover, its T-ideal of identities is generated by $[x_1,x_2,x_3]$ (cf. [9]).

Our aim here is to find explicit generators of $(K\langle x_{ij}\rangle/I)^{Gl_n}$ in the simplest noncommutative case, namely when $n=2$ and $I=I(E)$ is the T-ideal of identities of E (i.e., I is the ideal of $K\langle x_{ij}\rangle$ generated by the elements $[u_1,u_2,u_3]$ as u_k vary in $K\langle x_{ij}\rangle$). We note that in the commutative case $K[x_{ij}]^{Gl_n}$ is generated by traces of powers of X, and it is sufficient to take $Tr(X), Tr(X^2),\dots,Tr(X^n)$, since $Tr(X^{n+1})$ can be expressed by traces of lower powers using the Cayley–Hamilton theorem. We will show that the situation for $(K\langle x_{11},x_{12},x_{21},x_{22}\rangle/I(E))^{Gl_2}$ is completely analogous.

Let us mention that a related problem is studied in [2]. Take generic matrices $Y_r=(y_{ij}^{(r)})$, $Z_r=(z_{ij}^{(r)})$ $(r=1,2,\dots)$, where $F=K[y_{ij}^{(r)},z_{ij}^{(r)} \mid 1\le i,j\le n, r=1,2,\dots]$ is a free supercommutative superalgebra. The algebra $K\langle Y_r,Z_r \mid r=1,2,\dots\rangle$ is a universal algebra in the variety of superalgebras defined by $E^{n\times n}$. The conjugation action of Gl_n on matrices induces an action on F, and it is shown in [2] that F^{Gl_n} is the trace algebra (what is called supertace algebra there) of $K\langle Y_r,Z_r\rangle$.

For simplicity let us introduce the notation

$$R=K\langle x_{11},x_{12},x_{21},x_{22}\rangle/I(E).$$

Another fact we do not use but seems worth mentioning that it is proved in [11] that any algebra satisfying the identity $[x,y,z]=0$ can be embedded into a supercommutative superalgebra. This applies for the algebra R.

The results of [4] imply that the graded algebra R^{Gl_2} has a rational Hilbert series, and now we compute it explicitly.

Lemma 2.1. *The Hilbert series of R^{Gl_2} is*

$$H(R^{Gl_2};t)=\frac{1+2t^3+t^4}{(1-t)(1-t^2)}.$$

Proof. It is known that the four variable Hilbert series of R is

$$H(R;t_1,t_2,t_3,t_4)=\frac{1+\sum_{1\le i<j\le 4} t_it_j+t_1t_2t_3t_4}{\prod_{k=1}^4(1-t_k)}$$

(see for example [5]). The subgroup Sl_2 of Gl_2 has the same algebra of invariants for this action, so we may apply Weyl's unitarian trick. The special unitarian group SU_2 is a maximal compact subgroup of Sl_2, and it has the same algebra of invariants. So similar to the free algebra case treated in [1], a noncommutative Molien–Weyl formula is valid:

$$H(R^{Sl_2};t)=\int_{SU_2} H(R;\rho_1(g)t,\rho_2(g)t,\rho_3(g)t,\rho_4(g)t)\,\mathrm{d}\mu \quad \text{for } |t|<1,$$

where μ a normalized Haar measure and $\rho_1(g)$, $\rho_2(g)$, $\rho_3(g)$, $\rho_4(g)$ are the eigenvalues of the image of $g\in SU_2$ in the given four dimensional representation on $Span_K\{x_{11},x_{12},x_{21},x_{22}\}$. (Analogous Molien–Weyl formula for the Hilbert series of the algebra of invariants of a finite group acting on a relatively free algebra can be found in [6].)

The subgroup

$$T=\left\{\begin{pmatrix} z & 0\\ 0 & z^{-1}\end{pmatrix}\,\middle|\, |z|=1\right\}$$

is a maximal torus in SU_2, and the eigenvalues of the image of

$$\begin{pmatrix} z & 0\\ 0 & z^{-1}\end{pmatrix}$$

in the given four-dimensional representation are $1,z^2,z^{-2},1$. The Haar measure on T is $(1/2\pi\mathrm{i})z^{-1}\,\mathrm{d}z$, the Weyl group of SU_2 has two elements, and the only positive root $\theta : T\to\mathbb{C}$ is given by

$$\theta\left(\begin{pmatrix} z & 0\\ 0 & z^{-1}\end{pmatrix}\right)=z^2.$$

Hence by the Weyl integration formula the above expression is equal to the following complex integral over the unit circle:

$$\frac{1}{2}\int_{|z|=1}\frac{(2-z^2-z^{-2})(1+2t^2(1+z^2+z^{-2})+t^4)}{(1-t)^2(1-tz^2)(1-tz^{-2})}\frac{1}{2\pi\mathrm{i}}z^{-1}\,\mathrm{d}z \quad (|t|<1).$$

The above integral can be written as a sum

$$(1+t^4)A+\frac{t^2}{(1-t)^2}(B_1+B_{-1}),$$

where

$$B_j=\frac{1}{2\pi\mathrm{i}}\int_{|z|=1}\frac{(1-z^{2j})(1+z^2+z^{-2})}{z(1-tz^2)(1-tz^{-2})}\,\mathrm{d}z \quad (j=\pm 1).$$

Observe that by the commutative Molien–Weyl formula A is just the Hilbert series of the algebra of invariants $K[x_{11},x_{12},x_{21},x_{22}]^{Gl_2}$, and it is well known that this latter algebra is generated by the trace and the determinant, so

$$A=\frac{1}{(1-t)(1-t^2)}.$$

The substitution $z \to z^{-1}$ shows that $B_1 = B_{-1}$, and one can evaluate B_1 using the residue theorem. For any $0 < |t| < 1$ the function

$$\frac{P(z)}{Q(z)} = \frac{(1-z^2)(z^4+z^2+1)}{z(1-tz^2)(z^2-t)}$$

has three poles in the unit circle at 0 and $\pm\sqrt{t}$; hence by the residue theorem

$$B_1 = Res\frac{P(z)}{Q(z)}\Big|_{z=0} + Res\frac{P(z)}{Q(z)}\Big|_{z=\sqrt{t}} + Res\frac{P(z)}{Q(z)}\Big|_{z=-\sqrt{t}},$$

and since the multiplicity of the roots of $Q(z)$ is one, for any root η of $Q(z)$ we have

$$Res\frac{P(z)}{Q(z)}\Big|_{z=\eta} = \frac{P(\eta)}{Q'(\eta)}$$

(where Q' is the derivative of Q,) thus we get

$$B_1 = \frac{1}{-t} + 2\frac{(1-t)(t^2+t+1)}{\sqrt{t}(1-t^2)2\sqrt{t}} = \frac{t}{1+t},$$

and the claim follows. (Let us note that in the case of rational representations of Sl_n, a purely algebraic approach to the Molien–Weyl formula, the Weyl integration formula and the residue calculus is given in [1]. It is also possible to compute the above Hilbert series without referring to integrals, by computing the irreducible decomposition of the representations of Sl_2 on the homogeneous components of R.) □

Theorem 2.2. *R^{Gl_2} is generated by $Tr(X)$, $Tr(X^2)$, and $Tr(X^3)$.*

Proof. For a survey on computational techniques in relatively free algebras we refer to [5]. An account on the structure of the relatively free algebras defined by $I(E)$ can be found also in [3].

Denote by C the *commutator ideal* of R, that is, we have

$$R/C \cong K[x_{11}, x_{12}, x_{21}, x_{22}].$$

C^i/C^{i+1} has a natural R/C-module structure for $i = 0, 1, \ldots$. Since $C^3 = 0$, it suffices to find elements in R^{Gl_2} such that their images generate $(R/C)^{Gl_2}$, and $(C/C^2)^{Gl_2}$ and $(C^2)^{Gl_2}$ as an $(R/C)^{Gl_2}$-module. (See [4] for more general applications of this idea.)

The traces of powers of X are obviously invariants in the free case, hence $Tr(X^i)$ is contained in R^{Gl_2} for any i. It is a well known fact that $Tr(X)$, $Tr(X^2)$ generate $(R/C)^{Gl_2}$.

It is also easy to understand $(C^2)^{Gl_2}$. Recall that a polynomial is called *proper* if it is a linear combination of products of long commutators. There is only one proper polynomial in C^2 up to a scalar multiple, namely

$$\begin{aligned}[x_{11}, x_{22}][x_{12}, x_{21}] &= -[x_{11}, x_{12}][x_{22}, x_{21}] \\ &= [x_{11}, x_{21}][x_{22}, x_{12}] = \tfrac{1}{6}S_4(x_{11}, x_{12}, x_{21}, x_{22}),\end{aligned}$$

where

$$S_4(x_1,x_2,x_3,x_4)=\sum_{\pi\in Sym(4)} sign(\pi)x_{\pi(1)}x_{\pi(2)}x_{\pi(3)}x_{\pi(4)}$$

is the *standard polynomial* of degree four. Hence C^2 is a free R/C-module of rank one generated by $S_4(x_{11},x_{12},x_{21},x_{22})$. The polynomial $S_4(x_{11},x_{12},x_{21},x_{22})$ generates a one dimensional irreducible Gl_4-submodule of $K\langle x_{11},x_{12},x_{21},x_{22}\rangle$, where Gl_4 acts by linear substitution of the variables, and this representation is equivalent with $det(-)$. Since the representation of Gl_2 on $K\langle x_{ij}\rangle$ given by conjugation of X maps the elements of Gl_2 to determinant 1 elements of Gl_4, we have that $S_4(x_{11},x_{12},x_{21},x_{22})$ is contained in $K\langle x_{ij}\rangle^{Gl_2}$, hence it is clearly contained in R^{Gl_2}. So we have that $(C^2)^{Gl_2}$ is the free $(R/C)^{Gl_2}$-submodule of C^2 generated by $S_4(x_{11},x_{12},x_{21},x_{22})$.

The only piece left is C/C^2. It is a free R/C-module generated by

$$h_1=[x_{21},x_{12}],\quad h_2=[x_{22},x_{11}],\quad h_3=[x_{12},x_{22}],$$
$$h_4=[x_{21},x_{11}],\quad h_5=[x_{12},x_{11}],\quad h_6=[x_{21},x_{22}].$$

Hence its multigraded Hilbert series is

$$H(C/C^2;t_1,t_2,t_3,t_4)=\frac{\sum_{1\le i<j\le 4} t_it_j}{\prod_{k=1}^{4}(1-t_k)}.$$

We can compute the Hilbert series of $(C/C^2)^{Gl_2}$ by the same method as in Lemma 2.1, and we get that

$$H((C/C^2)^{Gl_2};t)=\frac{2t^3}{(1-t)(1-t^2)}.$$

The Hilbert series shows that we have to find two invariants of degree 3 whose images generate a free $(R/C)^{Gl_2}$-submodule of C/C^2. One of them is obviously the commutator

$$g_1=[Tr(X^2),Tr(X)].$$

We note that the results of [3] imply that $Tr(X)$, $Tr(X^2)$ generate a subalgebra of R which is isomorphic to the factor algebra of the free algebra of rank two modulo the T-ideal of identities of E. In particular, g_1 is nonzero, and so it is not even contained in C^2. To find the other recall that $Tr(X^3)$ can be expressed by $Tr(X)$ and $Tr(X^2)$ modulo the commutator ideal, using the commutative Cayley–Hamilton theorem. Thus we get that

$$g_2=Tr(X^3)-\tfrac{3}{2}Tr(X)Tr(X^2)+\tfrac{1}{2}Tr^3(X)$$

is also contained in C. Denote by $\bar{g}_i$ the images of g_i in C/C^2. We claim that

$$\bar{g}_1\in\sum_{k=2}^{6} R/C\cdot h_k$$

and

$$\bar{g}_2 \in \tfrac{1}{2}(x_{22} - x_{11})h_1 + \sum_{k=2}^{6} R/C \cdot h_k.$$

To show the claim consider the K-linear basis

$$\{x_{11}^{\alpha_1}x_{22}^{\alpha_2}x_{12}^{\alpha_3}x_{21}^{\alpha_4}h_i^{\beta} \mid i=1,\dots,6,\ \beta=0,1,\ \alpha_1,\alpha_2,\alpha_3,\alpha_4 \geq 0\}$$

of R/C^2. We order the variables $x_{11}<x_{22}<x_{12}<x_{21}$. For an arbitrary monomial $y_1\dots y_r \in R/C^2$ denote by $\tau(y_1\dots y_r)$ the monomial with the same multidegree in which the variables appear in nondecreasing order. The normal form of an arbitrary monomial $y_1\dots y_s \in R/C^2$ with respect to the above basis is

$$y_1\dots y_s = \tau(y_1\dots y_s) + \sum_{i\langle j, y_i\rangle y_j} \tau(y_1\dots y_{i-1}y_{i+1}\dots y_{j-1}y_{j+1}\dots y_s)[y_i, y_j].$$

Hence to compute the coefficient of $h_1=[x_{21},x_{12}]$ in $\bar{g}_1$ we have to consider the monomials of $[Tr(X^2), Tr(X)]$ in which x_{12}, x_{21} appear in the reverse order. $Tr(X)$ does not contain these variables, but $Tr(X^2)=x_{11}^2+x_{22}^2+x_{12}x_{21}+x_{21}x_{12}$ contains the term $x_{21}x_{12}$. So the coefficient of h_1 in the normal form of $\bar{g}_1$ is the same as in $[x_{21}x_{12}, Tr(X)]$, and the normal form of this latter polynomial is $x_{21}h_5+x_{12}h_4+x_{21}h_3+x_{12}h_6$. We can determine the coefficient of h_1 in the normal form of $\bar{g}_2$ similarly. It is easy to see that

$$\begin{aligned} g_2 &= x_{21}x_{12}x_{22}+x_{21}x_{11}x_{12}+x_{22}x_{21}x_{12}-\tfrac{3}{2}Tr(X)x_{21}x_{12} \\ &\quad + \text{terms in which } x_{21} \text{ and } x_{12} \text{ do not appear in this order.} \end{aligned}$$

Hence the coefficient of $x_{22}[x_{21},x_{12}]$ in $\bar{g}_2$ is $1+1-\frac{3}{2}=\frac{1}{2}$ and the coefficient of $x_{11}[x_{21},x_{12}]$ is $1-\frac{3}{2}=-\frac{1}{2}$, as we claimed. Now since C/C^2 is a free R/C-module generated by $h_1,\dots,h_6$, and the coefficient of h_1 in $\bar{g}_1$ is zero while in $\bar{g}_2$ it is nonzero, we have that $\bar{g}_1$ and $\bar{g}_2$ generate a free R/C-submodule in C/C^2, implying that they generate a free $(R/C)^{Gl_2}$-submodule in $(C/C^2)^{Gl_2}$. The Hilbert series of $(C/C^2)^{Gl_2}$ shows that these are all the invariants in C/C^2.

To finish the proof we have to express $S_4(x_{11},x_{12},x_{21},x_{22})$ by $Tr(X)$, $Tr(X^2)$, and $Tr(X^3)$. Let us observe that since commutators are central in R and g_2 is contained in C, we have that $[g_2, Tr(X)]$ is contained in C^2. On the other hand, it is of degree four. Hence it must be a scalar multiple of $S_4(x_{11},x_{12},x_{21},x_{22})$. To determine the scalar let us substitute

$$X \to \begin{pmatrix} v_1 & v_2 \\ v_3 & v_4 \end{pmatrix} = V,$$

where v_1,v_2,v_3,v_4 are among the generators of E, that is, $v_iv_j+v_jv_i=0$ $(1\leq i,j\leq 4)$. One can compute that $Tr(V)=v_1+v_4$, $Tr(V^2)=0$, and $Tr(V^3)=3(v_1v_2v_3-v_2v_3v_4)$, hence

$$[Tr(V^3)-\tfrac{3}{2}Tr(V)Tr(V^2)+\tfrac{1}{2}Tr^3(V), Tr(V)] = 12v_1v_2v_3v_4.$$

Since $S_4(v_1,v_2,v_3,v_4)=24v_1v_2v_3v_4$, we have the equality

$$S_4(x_{11},x_{12},x_{21},x_{22})=[2Tr(X^3)-3Tr(X)Tr(X^2),Tr(X)]. \qquad \square$$

3. The Cayley–Hamilton theorem for 2×2 matrices over E

We keep the notation $R=K\langle x_{11},x_{12},x_{21},x_{22}\rangle/I(E)$. In view of Corollary 1.2 and Theorem 2.2 we know that the coefficients of $\chi^{(2)}(t)\in R[t]$ can be expressed by $Tr(X)$, $Tr(X^2)$, and $Tr(X^3)$, so let us look at the characteristic polynomial $\chi^{(2)}(t)$ of

$$X=\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22}\end{pmatrix}$$

more closely. Recall that

$$X_1=adj(X)=\begin{pmatrix} x_{22} & -x_{12} \\ -x_{21} & x_{11}\end{pmatrix},$$

and direct computation shows that $XX_1=aI+B$, where

$$a=\tfrac{1}{2}(x_{11}\circ x_{22}-x_{12}\circ x_{21})=\tfrac{1}{2}(Tr^2(X)-Tr(X^2))$$

(we use the notation $x\circ y=xy+yx$) and

$$B=\begin{pmatrix} \frac{1}{2}([x_{11},x_{22}]-[x_{12},x_{21}]) & [x_{12},x_{11}] \\ [x_{21},x_{22}] & -\frac{1}{2}([x_{11},x_{22}]-[x_{12},x_{21}])\end{pmatrix}.$$

It is easy to see that

$$X_2=adj(XX_1)=adj(aI+B)=aI-B,$$

and

$$XX_1X_2=a^2I-B^2,$$

implying that

$$\begin{aligned} det^{(2)}(X)&=\tfrac{1}{4}(Tr^2(X)-Tr(X^2))^2-\tfrac{1}{4}([x_{11},x_{22}]-[x_{12},x_{21}])^2-[x_{12},x_{11}][x_{21},x_{22}]\\ &=\tfrac{1}{4}((Tr^2(X)-Tr(X^2))^2+S_4(x_{11},x_{12},x_{21},x_{22})).\end{aligned}$$

By definition the characteristic polynomial is

$$\begin{aligned}\chi^{(2)}(t)&=det^{(2)}(X-tI)\\ &=\tfrac{1}{4}((Tr^2(X-tI)-Tr((X-tI)^2))^2+S_4(x_{11}-t,x_{12},x_{21},x_{22}-t)\\ &=t^4-2Tr(X)t^3+(2Tr^2(X)-Tr(X^2))t^2\\ &\quad+(\tfrac{1}{2}Tr(X)\circ Tr(X^2)-Tr^3(X))t+det^{(2)}(X).\end{aligned}$$

Using the expression for $S_4(x_{11},x_{12},x_{21},x_{22})$ as a function of $Tr(X^i)$ $(i=1,2,3)$ obtained in the proof of Theorem 2.2 we get that

$$det^{(2)}(X)=\tfrac{1}{4}(Tr^4(X)+Tr^2(X^2)+\tfrac{1}{2}Tr^2(X)Tr(X^2)$$
$$-\tfrac{5}{2}Tr(X^2)Tr^2(X)+2[Tr(X^3),Tr(X)]).$$

A similar computation yields the second left determinant of X, and the theorem of Szigeti from [10] obtains the following nice compact form in this special case:

Theorem 3.1. *For any 2×2 matrix A over a K-algebra S satisfying the identity $[x_1,x_2,x_3]=0$ we have that*

$$A^4-2A^3Tr(A)+A^2(2Tr^2(A)-Tr(A^2))+A(\tfrac{1}{2}Tr(A)\circ Tr(A^2)-Tr^3(A))$$
$$+\tfrac{1}{4}(Tr^4(A)+Tr^2(A^2)+\tfrac{1}{2}Tr^2(A)Tr(A^2)-\tfrac{5}{2}Tr(A^2)Tr^2(A)+2[Tr(A^3),Tr(A)])\cdot I$$

and

$$A^4-2Tr(A)A^3+(2Tr^2(A)-Tr(A^2))A^2+(\tfrac{1}{2}Tr(A)\circ Tr(A^2)-Tr^3(A))A$$
$$+\tfrac{1}{4}(Tr^4(A)+Tr^2(A^2)-\tfrac{5}{2}Tr^2(A)Tr(A^2)+\tfrac{1}{2}Tr(A^2)Tr^2(A)-2[Tr(A^3),Tr(A)])\cdot I$$

are equal to zero in $S^{2\times 2}$.

Remark. For comparison we mention that certain Cayley–Hamilton equations for the matrix superalgebras are defined and studied in [7].

Corollary 3.2. *For any 2×2 matrix A over a K-algebra S satisfying the identity $[x_1,x_2,x_3]=0$ we have that if $Tr(A)=Tr(A^2)=0$, then $A^4=0$.*

Proof. Observe that in the Cayley–Hamilton equation of Theorem 3.1 $Tr(A^3)$ does not appear in the coefficient of A, it appears only in the constant term in the commutator $[Tr(A^3),Tr(A)]$. □

Taking traces of both sides in any of the equalities of Theorem 3.1 we get the following formula:

Corollary 3.3. *In R^{Gl_2} we have the equality*

$$Tr(X^4)=Tr(X^3)\circ Tr(X)+\tfrac{1}{2}Tr^2(X^2)+\tfrac{1}{2}Tr^4(X)-2Tr(X)Tr(X^2)Tr(X).$$

Remark. Our results about R^{Gl_2} are very similar to those about $K[x_{ij}]^{Gl_2}$: the algebra of invariants is generated by powers of traces of X, and it suffices to take powers of order up to 3, because $Tr(X^4)$, $Tr(X^5)$, ... can be expressed by traces of powers of lower degree using the noncommutative Cayley–Hamilton theorem.

However, contrary to the commutative case, the coefficients of the characteristic polynomial do not generate R^{Gl_2}, as one can see it from the explicit form of the coefficients of the characteristic polynomial.

Conversely, we show that if for some T-ideal I the algebra $(K\langle x_{ij}\rangle/I)^{Gl_n}$ is generated by traces of powers of X, then $K\langle x_{ij}\rangle/I$ satisfies a polynomial identity of degree three. Indeed, consider the invariant $Tr(X^3)$. It contains the monomial $x_{11}x_{12}x_{21}$. Now exchange the first and the second variables in each monomial of $Tr(X^3)$. Since the action of $Sym(3)$ by place permutation on the degree three homogeneous polynomials in $K\langle x_{ij}\rangle$ commutes with the action of Gl_n, the resulting polynomial is also an invariant. On the other hand, it contains the monomial $x_{12}x_{11}x_{21}$. It is obvious that products of traces of powers of X do not contain this monomial, so if this invariant can be expressed by traces of powers of X, then I must contain a degree three polynomial. (We note that if a unitary algebra satisfies a degree 3 identity, then it satisfies the identity $[x_1,x_2,x_3]=0$.)

The question whether $(K\langle x_{ij}\rangle/I(E))^{Gl_n}$ is generated by traces of powers of X remains open for $n\geq 3$.

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